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Research Article

Precise Asymptotics in the Law of Iterated Logarithm for Moving Average Process under Dependence

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Let $\{\xi_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed and ϕ -mixing random variables, and let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. In this paper, we get precise asymptotics in the law of the logarithm for linear process $\{X_k = \sum_{i=-\infty}^{+\infty} a_{i+k}\xi_i, k \geq 1\}$, which extend Liu and Lin's (2006) result to moving average process under dependence assumption.

1. Introduction and Main Results

Let $\{\xi_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed random variables with zero means and finite variances, and let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. Let

$$X_k = \sum_{i=-\infty}^{+\infty} a_{i+k}\xi_i, \quad k \geq 1, \quad (1.1)$$

be the moving average process based on $\{\xi_i, -\infty < i < \infty\}$. As usual, we denote $S_n = \sum_{k=1}^n X_k$, $n \geq 1$ as the sequence of partial sums.

Under the assumption that $\{\xi_i, -\infty < i < \infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained. Ibragimov [1] established the central limit theorem; Burton and Dehling [2] obtained a large deviation principle; Yang [3] established the central limit theorem and the law of the iterated logarithm;

Li et al. [4] obtained the complete convergence result for $\{X_k, k \geq 1\}$. As we know, X_k ($k \geq 1$) are dependent even if $\{\xi_i, -\infty < i < \infty\}$ is a sequence of i.i.d. random variables. Therefore, we introduce the definition of ϕ -mixing,

$$\phi(m) := \sup_{k \geq 1} \left\{ |P(B | A) - P(B)|, A \in \mathcal{F}_{-\infty}^k, P(A) \neq 0, B \in \mathcal{F}_{k+m}^\infty \right\} \rightarrow 0, \quad m \rightarrow \infty, \quad (1.2)$$

where $\mathcal{F}_a^b = \sigma(\xi_i, a \leq i \leq b)$. Many limiting results of moving average for ϕ -mixing have been obtained. For example, Zhang [5] got complete convergence.

Theorem A. Suppose that $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ -mixing random variables with $\sum_{m=1}^\infty \phi^{1/2}(m) < \infty$, and $\{X_k, k \geq 1\}$ is defined as (1.1). Let $h(x) > 0$ ($x > 0$) be a slowly varying function and $1 \leq t < 2$, $r \geq 1$, then $E\xi_1 = 0$ and $E|\xi_1|^{rt}h(|Y_1^t|) < \infty$ imply

$$\sum_{n=1}^\infty n^{r-2}h(n)P(|S_n| \geq n^{1/t}\epsilon) < \infty, \quad \forall \epsilon > 0. \quad (1.3)$$

Li and Zhang [6] achieved precise asymptotics in the law of the iterated logarithm.

Theorem B. Suppose that $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ -mixing random variables with mean zeros and finite variances, $\sum_{m=1}^\infty \phi^{1/2}(m) < \infty$, and $0 < \sigma^2 = E\xi_1^2 + 2 \sum_{k=2}^\infty E\xi_1\xi_k < \infty$, $E\xi_1^2(\log^+|\xi_1|)^{\delta-1} < \infty$, for $\delta > 0$. Suppose that $\{X, X_k, k \geq 1\}$ is defined as in (1.1), where $\{a_i, -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^\infty |a_i| < \infty$, then one has

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n=2}^\infty \frac{(\log n)^\delta}{n} P(|S_n| \geq \sqrt{n \log n \tau \epsilon}) = \frac{\tau^{2\delta+2}}{\delta+1} E|N|^{2\delta+2}, \quad (1.4)$$

where $\tau =: \sigma \sum_{i=-\infty}^\infty a_i$, N is a standard normal random variable.

On the other hand, since Hsu and Robbins [7] introduced the concept of the complete convergence, there have been extensions in some directions. For the case of i.i.d. random variables, Davis [8] proved $\sum_{n=1}^\infty (\log n/n)P(|S_n| \geq \sqrt{n \log n \epsilon}) < \infty$, for $\epsilon > 0$ if and only if $EX_1 = 0$, $EX_1^2 < \infty$. Gut and Spătaru [9] gave the precise asymptotics of $\sum_{n=2}^\infty ((\log n)^\delta/n)P(|S_n| \geq \sqrt{n \log n \epsilon})$. We know that complete convergence can be derived from complete moment convergence. Liu and Lin [10] introduced a new kind of convergence of $\sum_{n=2}^\infty ((\log n)^{\delta-1}/n^2)E|S_n|^2 I\{|S_n| \geq \sqrt{n \log n \epsilon}\}$. In this note, we show that the precise asymptotics for the moment convergence hold for moving-average process when $\{\xi_i, -\infty < i < \infty\}$ is a strictly stationary ϕ -mixing sequences. Now, we state the main results.

Theorem 1.1. Suppose that $\{X, X_k, k \geq 1\}$ is defined as in (1.1), where $\{a_i, -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^\infty |a_i| < \infty$, and $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically

distributed ϕ -mixing random variables with mean zeros and finite variances, $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$ and $0 < \sigma^2 = E\xi_1^2 + 2 \sum_{k=2}^{\infty} E\xi_1 \xi_k < \infty$, $E\xi_1^2 (\log^+ |\xi_1|)^{\delta} < \infty$, for $0 < \delta \leq 1$, then one has

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} E|S_n|^2 I \left\{ |S_n| \geq \sqrt{n \log n \tau \epsilon} \right\} = \frac{\tau^{2\delta+2}}{\delta} E|N|^{2\delta+2}, \quad (1.5)$$

where $\tau =: \sigma \sum_{i=-\infty}^{\infty} a_i$.

Theorem 1.2. Under the conditions in Theorem 1.1, one has

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta-1}}{n^2 \log n} E S_n^2 I \left\{ |S_n| \geq \sqrt{n \log \log n \tau \epsilon} \right\} = \frac{\tau^{2\delta+2} E|N|^{2(\delta+1)}}{\delta}. \quad (1.6)$$

Remark 1.3. In this paper, we generate the results of Liu and Lin [10] to linear process under dependence based on Theorem B by using the technique of dealing with the innovation process in Zhang [5].

We first proceed with some useful lemmas.

Lemma 1.4. Let $\{X_k, k \geq 1\}$ be defined as in (1.1), and let $\{\xi_i, -\infty < i < \infty\}$ be a sequence of identically distributed ϕ -mixing random variables with $E\xi_1 = 0$, $E\xi_1^2 < \infty$, $0 < \sigma^2 = E\xi_1^2 + 2 \sum_{k=2}^{\infty} E\xi_1 \xi_k < \infty$, $\sum_{m=1}^{\infty} \phi^{1/2}(2^m) < \infty$, then

$$\frac{S_n}{\tau \sqrt{n}} \xrightarrow{D} N(0, 1). \quad (1.7)$$

The proof is similar to Theorem 1 in [11]. Set $\Delta_n = \sup_x |P(|S_n| \geq \sqrt{n}x) - P(|N| \geq x)|$. From Lemma 1.4, one can get $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 1.5 (see [2]). Let $\sum_{i=-\infty}^{+\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{+\infty} a_i$ and $k \geq 1$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=-\infty}^{+\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k. \quad (1.8)$$

Lemma 1.6 (see [12]). Let $\{X_i, i \geq 1\}$ be a sequence of ϕ -mixing random variables with zero means and finite second moments. Let $S_n = \sum_{i=1}^n X_i$. If exists C_n such that $\max_{1 \leq i \leq n} ES_n^2 \leq C_n$, then for all $q \geq 2$, there exists $C = C(q, \phi(\cdot))$ such that

$$E \max_{1 \leq i \leq n} |S_i|^q \leq C \left(C_n^{q/2} + E \max_{1 \leq i \leq n} |X_i|^q \right). \quad (1.9)$$

2. Proofs

Proof of Theorem 1.1. Without loss of generality, we assume that $\tau = 1$. We have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I \left\{ |S_n| \geq \sqrt{n \log n \epsilon} \right\} \\ &= \epsilon^2 \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P \left\{ |S_n| \geq \sqrt{n \log n \epsilon} \right\} + \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx. \end{aligned} \quad (2.1)$$

Set $d(\epsilon) = \exp(M\epsilon^{-2})$, where $M > 1$. By Theorem B, we need to show

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx = \frac{1}{\delta(\delta+1)} E|N|^{2\delta+2}. \quad (2.2)$$

By Proposition 5.1 in [10], we have

$$\lim_{\epsilon \searrow 0} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P \left(|N| \geq \frac{x}{\sqrt{n}} \right) dx = \frac{1}{\delta(\delta+1)} E|N|^{2\delta+2}. \quad (2.3)$$

Hence, Theorem 1.1 will be proved if we show the following two propositions. \square

Proposition 2.1. *One has*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \left| \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx - \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P \left(|N| \geq \frac{x}{\sqrt{n}} \right) dx \right| = 0. \quad (2.4)$$

Proof. Write

$$\begin{aligned} & \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \left| \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx - \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P \left(|N| \geq \frac{x}{\sqrt{n}} \right) dx \right| \\ & \leq \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n} \log n \\ & \quad \times \int_0^{\infty} 2(x + \epsilon) \left| P \left(|S_n| \geq \sqrt{n \log n (x + \epsilon)} \right) - P \left(|N| \geq \sqrt{\log n (x + \epsilon)} \right) \right| dx \\ & \leq \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n} (\Delta_{n1} + \Delta_{n2} + \Delta_{n3}), \end{aligned} \quad (2.5)$$

where

$$\begin{aligned}\Delta_{n1} &= \log n \int_0^{1/\sqrt{\log n} \Delta_n^{1/4}} 2(x + \epsilon) \left| P\left(|S_n| \geq \sqrt{n \log n}(x + \epsilon)\right) - P\left(|N| \geq \sqrt{\log n}(x + \epsilon)\right) \right| dx, \\ \Delta_{n2} &= \log n \int_{1/\sqrt{\log n} \Delta_n^{1/4}}^{\infty} 2(x + \epsilon) P\left(|S_n| \geq \sqrt{n \log n}(x + \epsilon)\right) dx, \\ \Delta_{n3} &= \log n \int_{1/\sqrt{\log n} \Delta_n^{1/4}}^{\infty} 2(x + \epsilon) P\left(|N| \geq \sqrt{\log n}(x + \epsilon)\right) dx.\end{aligned}\tag{2.6}$$

Since $n \leq d(\epsilon)$ implies $\sqrt{\log n \epsilon} \leq \sqrt{M}$, we have

$$\Delta_{n1} \leq C \log n \Delta_n \left(\frac{1}{\sqrt{\log n} \Delta_n^{1/4}} + \epsilon \right)^2 \leq C \left(\Delta_n^{1/4} + \sqrt{M \Delta_n} \right)^2.\tag{2.7}$$

For Δ_{n3} , by Markov's inequality, we get

$$\Delta_{n3} \leq C \log n \int_{1/\sqrt{\log n} \Delta_n^{1/4}}^{\infty} \frac{1}{(\log n)^{3/2} (x + \epsilon)^2} dx \leq C \Delta_n^{1/4}.\tag{2.8}$$

From (2.7) and (2.8), we can get

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n} (\Delta_{n1} + \Delta_{n3}) = 0.\tag{2.9}$$

Note that $\sum_{k=1}^n X_k = \sum_{i=-\infty}^{\infty} \sum_{k=1}^n a_{k+i} \xi_i = \sum_{i=-\infty}^{\infty} a_{ni} \xi_i$, where $a_{ni} = \sum_{k=1}^n a_{k+i}$. By Lemma 1.5, we can assume that

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^t \leq n, \quad t \geq 1, \quad \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \leq 1.\tag{2.10}$$

Set $S'_n = \sum_{i=-\infty}^{\infty} a_{ni} \xi_i I\{|a_{ni} \xi_i| \leq \sqrt{n \log n}(x + \epsilon)\}$. As $E \xi_1 = 0$, by (2.10), we have

$$\begin{aligned}|ES'_n| &\leq C \left| E \sum_{i=-\infty}^{\infty} a_{ni} \xi_i I\{|a_{ni} \xi_i| > \sqrt{n \log n}(x + \epsilon)\} \right| \\ &\leq C \sum_{i=-\infty}^{\infty} |a_{ni}| E|\xi_1| I\{|a_{ni} \xi_1| > \sqrt{n \log n}(x + \epsilon)\} \\ &\leq C n E|\xi_1| I\{\tilde{a} |\xi_1| > \sqrt{n \log n}(x + \epsilon)\}\end{aligned}$$

$$\begin{aligned}
&\leq CnE|\xi_1|I\left\{|\xi_1| > \sqrt{n\log n}(x+\epsilon)\right\} \\
&\leq Cn\left(E\xi_1^2\right)^{1/2}\left(P\left(|\xi_1| > \sqrt{n\log n}(x+\epsilon)\right)\right)^{1/2} \\
&\leq C\frac{\sqrt{n}E\xi_1^2}{\sqrt{\log n}(x+\epsilon)}.
\end{aligned} \tag{2.11}$$

So, when $x \in (1/\sqrt{\log n}\Delta_n^{1/4}, \infty)$,

$$\frac{|ES'_n|}{\sqrt{n\log n}(x+\epsilon)} \leq C\frac{E\xi_1^2}{\log n\left(1/\sqrt{\log n}\Delta_n^{1/4}+\epsilon\right)^2} < \epsilon, \quad \text{for } n \text{ large enough.} \tag{2.12}$$

By (2.12), we have

$$\begin{aligned}
&\sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n} \Delta_{n2} \\
&\leq \sum_{n=2}^{d(\epsilon)} \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} \frac{(\log n)^{\delta}(x+\epsilon)}{n} \\
&\quad \times \left[P\left(\sup_i |a_{ni}\xi_i| > \sqrt{n\log n}(x+\epsilon)\right) + P\left(|S'_n - ES'_n| \geq \frac{\sqrt{n\log n}(x+\epsilon)}{2}\right) \right] dx \\
&=: H_1 + H_2.
\end{aligned} \tag{2.13}$$

Set $I_{nj} = \{j \in \mathcal{L}, 1/(j+1) < |a_{nj}| \leq 1/j, j = 1, 2, \dots\}$, then $\bigcup_{j \geq 1} I_{nj} = \mathcal{L}$ (referred by [4]). We can get

$$\sum_{j=1}^k \#I_{nj} \leq n(k+1). \tag{2.14}$$

Then,

$$\begin{aligned}
&P\left\{\sup_i |a_{ni}\xi_i| > \sqrt{n\log n}(x+\epsilon)\right\} \\
&\leq \sum_{i=-\infty}^{\infty} P\left\{|a_{ni}\xi_i| > \sqrt{n\log n}(x+\epsilon)\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{\infty} \sum_{i \in I_{n_j}} P \left\{ |\xi_1| \geq \sqrt{n \log n j} (x + \epsilon) \right\} \leq \sum_{j=1}^{\infty} (\#I_{n_j}) P \left\{ |\xi_1| \geq \sqrt{n \log n j} (x + \epsilon) \right\} \\
&\leq \sum_{j=1}^{\infty} \sum_{k \geq j} (\#I_{n_j}) P \left\{ \sqrt{n \log n k} (x + \epsilon) \leq |\xi_1| < \sqrt{n \log n (k+1)} (x + \epsilon) \right\} \\
&\leq \sum_{k=1}^{\infty} \sum_{j=1}^k (\#I_{n_j}) P \left\{ \sqrt{n \log n k} (x + \epsilon) \leq |\xi_1| < \sqrt{n \log n (k+1)} (x + \epsilon) \right\} \\
&\leq \sum_{k=1}^{\infty} n(k+1) P \left\{ \sqrt{n \log n k} (x + \epsilon) \leq |\xi_1| < \sqrt{n \log n (k+1)} (x + \epsilon) \right\} \\
&\leq \frac{\sqrt{n} E|\xi_1| I \left\{ |\xi_1| \geq \sqrt{n \log n} (x + \epsilon) \right\}}{\sqrt{\log n} (x + \epsilon)}.
\end{aligned} \tag{2.15}$$

So, we get

$$\begin{aligned}
H_1 &\leq C E|\xi_1| \int_{1/\sqrt{\log n} \Delta_n^{1/4}}^{\infty} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\quad \times \sum_{k=n}^{\infty} I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \\
&\leq C \int_0^{\infty} \sum_{k=2}^{\infty} E|\xi_1| I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \sum_{n=2}^k \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\leq C E \xi_1^2 \int_0^{\infty} (x + \epsilon)^{-1} \sum_{k=2}^{\infty} (\log k)^{\delta-1} I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \\
&\leq C E \xi_1^2 \int_0^{\infty} |\log^+ |\xi_1| - \log(x + \epsilon)|^{(\delta-1)} (x + \epsilon)^{-1} I \{ |\xi_1| > (x + \epsilon) \} dx \\
&\leq C E \xi_1^2 |\log^+ |\xi_1| - \log \epsilon|^{\delta} \leq C E \xi_1^2 (\log^+ |\xi_1|)^{\delta} + C E \xi_1^2 (-\log \epsilon)^{\delta}.
\end{aligned} \tag{2.16}$$

Therefore,

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_1 = 0. \tag{2.17}$$

By Lemma 1.6, noting that $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$, for $q > 2$,

$$\begin{aligned}
 H_2 &\leq C \sum_{n=2}^{d(\epsilon)} \int_0^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} (x+\epsilon)^{1-q} \\
 &\quad \times \left\{ \left(\sum_{i=-\infty}^{\infty} E(a_{ni}\xi_1)^2 I \left\{ |a_{ni}\xi_1| \leq \sqrt{n \log n} (x+\epsilon) \right\} \right)^{q/2} \right. \\
 &\quad + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I \left\{ |a_{ni}\xi_1| \leq \sqrt{n \log n \epsilon} \right\} \\
 &\quad \left. + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I \left\{ \sqrt{n \log n \epsilon} < |a_{ni}\xi_1| \leq \sqrt{n \log n} (x+\epsilon) \right\} \right\} dx \\
 &=: H_{21} + H_{22} + H_{23}.
 \end{aligned} \tag{2.18}$$

For H_{21} , we have

$$\begin{aligned}
 H_{21} &\leq \sum_{n=2}^{d(\epsilon)} \int_0^{\infty} \frac{(\log n)^{\delta-q/2}}{n} (x+\epsilon)^{1-q} \left(E\xi_1^2 I \left\{ |a_{ni}\xi_1| \leq \sqrt{n \log n} (x+\epsilon) \right\} \right)^{q/2} dx \\
 &\leq C\epsilon^{-2\delta} M^{\delta+1-q/2}.
 \end{aligned} \tag{2.19}$$

Then, for $0 < \delta \leq 1$, $q > 2$, we have

$$\lim_{M \rightarrow \infty} \limsup_{\epsilon \searrow 0} \epsilon^{2\delta} H_{21} = 0. \tag{2.20}$$

For H_{22} , we decompose it into two parts,

$$\begin{aligned}
 H_{22} &\leq \sum_{n=2}^{d(\epsilon)} \int_0^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} (x+\epsilon)^{1-q} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}|^q E|\xi_1|^q I \left\{ |a_{ni}\xi_1| \leq \sqrt{n \log n \epsilon} \right\} dx \\
 &\leq \epsilon^{2-q} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
 &\quad \times \sum_{j=1}^{\infty} (\#I_{nj}) j^{-q} \left\{ \sum_{k=0}^{2n} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \right. \\
 &\quad \left. + \sum_{k=2n+1}^{(j+1)n} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \right\} \\
 &=: H_{221} + H_{222}.
 \end{aligned} \tag{2.21}$$

It is easy to see that

$$\sum_{j=m}^{\infty} (\#I_{nj}) (j+1)^{-q} (m+1)^{q-1} \leq \sum_{j=1}^{\infty} (\#I_{nj}) (j+1)^{-1} \leq \sum_{i=-\infty}^{\infty} |a_{ni}| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \leq n. \quad (2.22)$$

So,

$$\sum_{j=m}^{\infty} (\#I_{nj}) j^{-q} \leq Cnm^{-(q-1)}. \quad (2.23)$$

Now, we estimate H_{221} , by (2.23),

$$\begin{aligned} H_{221} &\leq e^{2-q} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} \sum_{k=0}^{2n} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq e^{2-q} \sum_{k=2}^{d(\epsilon)} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \sum_{n=[k/2]}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} \\ &\leq e^{2-q} \sum_{k=2}^{d(\epsilon)} \frac{(\log k)^{\delta-q/2}}{k^{q/2-1}} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq \sum_{k=2}^{d(\epsilon)} (\log k)^{\delta-1} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq \sum_{k=2}^{d(\epsilon)} (\log k)^{-1} |\log^+ |\xi_1| - \log \epsilon|^\delta E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq CE\xi_1^2 (\log^+ |\xi_1|)^\delta + CE\xi_1^2 (-\log \epsilon)^\delta. \end{aligned} \quad (2.24)$$

For H_{222} , we have

$$\begin{aligned} H_{222} &\leq e^{2-q} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\ &\quad \times \sum_{k=2n+1}^{\infty} \sum_{j \geq k/n-1} (\#I_{nj}) j^{-q} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq e^{2-q} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{1-q/2}} \sum_{k=2n+1}^{\infty} k^{1-q} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq \sum_{k=2}^{d(\epsilon)} (\log k)^{\delta-1} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq \sum_{k=2}^{d(\epsilon)} (\log k)^{-1} |\log^+ |\xi_1| - \log \epsilon|^\delta E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\ &\leq CE\xi_1^2 (\log^+ |\xi_1|)^\delta + CE\xi_1^2 (-\log \epsilon)^\delta. \end{aligned} \quad (2.25)$$

From (2.24) and (2.25), we can get

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_{22} = \lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_{221} + \lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_{222} = 0. \quad (2.26)$$

Finally, $q > 2$, and we will get

$$\begin{aligned} H_{23} &\leq C \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} E|\xi_1|^q I \left\{ |\tilde{a}\xi_1| > \sqrt{n \log n \epsilon} \right\} \\ &\quad \times \int_0^\infty (x+\epsilon)^{1-q} \sum_{j=1}^\infty (\#I_{nj}) j^{-q} \left\{ \sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n} \right\} \\ &\quad \times I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1)(x+\epsilon)} \right\} dx \\ &\leq C \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} E|\xi_1|^q I \left\{ |\tilde{a}\xi_1| > \sqrt{n \log n \epsilon} \right\} \\ &\quad \times \int_0^\infty (x+\epsilon)^{1-q} I \left\{ |\xi_1| \leq \sqrt{2n \log(2n)(x+\epsilon)} \right\} dx + C \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-q/2}}{n^{1-q/2}} E|\xi_1|^q \\ &\quad \times \int_0^\infty \sum_{k=2n+1}^\infty \frac{k^{1-q}}{(x+\epsilon)^{q-1}} I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1)(x+\epsilon)} \right\} dx \\ &\leq C \sum_{k=2}^\infty E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} < |\xi_1| \leq \sqrt{(k+1) \log(k+1)\epsilon} \right\} \sum_{n=2}^k \frac{(\log n)^{\delta-1}}{n} \\ &\quad + CE\xi_1^2 \int_0^\infty (x+\epsilon)^{-1} \\ &\quad \times \sum_{k=4}^\infty (\log k)^{\delta-1} I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1)(x+\epsilon)} \right\} dx \\ &\leq C \sum_{k=2}^\infty (\log k)^\delta E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} < |\xi_1| \leq \sqrt{(k+1) \log(k+1)\epsilon} \right\} \\ &\quad + CE\xi_1^2 \int_0^\infty |\log^+ |\xi_1| - \log(x+\epsilon)|^{(\delta-1)} (x+\epsilon)^{-1} I\{|\xi_1| > (x+\epsilon)\} dx \\ &\leq CE\xi_1^2 |\log^+ |\xi_1| - \log \epsilon|^\delta \leq CE\xi_1^2 (\log^+ |\xi_1|)^\delta + CE\xi_1^2 (-\log \epsilon)^\delta, \end{aligned} \quad (2.27)$$

then

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_{23} = 0. \quad (2.28)$$

Hence, (2.4) can be referred from (2.9), (2.17), (2.20), (2.26), and (2.28). \square

Proposition 2.2. *One has*

$$\lim_{\epsilon \searrow 0} e^{2\delta} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \left| \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx - \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P\left(|N| \geq \frac{x}{\sqrt{n}}\right) dx \right| = 0. \quad (2.29)$$

Proof. Consider the following:

$$\begin{aligned} & \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \left| \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P(|S_n| \geq x) dx - \int_{\sqrt{n \log n \epsilon}}^{\infty} 2x P\left(|N| \geq \frac{x}{\sqrt{n}}\right) dx \right| \\ & \leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} 2(x + \epsilon) P\left(|N| \geq \sqrt{\log n}(x + \epsilon)\right) \\ & \quad + \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} 2(x + \epsilon) P\left(|S_n| \geq \sqrt{n \log n}(x + \epsilon)\right) dx \\ & =: G_1 + G_2. \end{aligned} \quad (2.30)$$

We first estimate G_1 , for $\theta > 2\delta$, by Markov's inequality,

$$\begin{aligned} G_1 & \leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} \frac{1}{(\log n)^{(\theta+2)/2} (x + \epsilon)^{\theta+1}} dx \\ & \leq CM^{\delta-\theta/2} \epsilon^{-2\delta}. \end{aligned} \quad (2.31)$$

Hence,

$$\lim_{M \rightarrow \infty} \limsup_{\epsilon \searrow 0} e^{2\delta} G_1 = 0. \quad (2.32)$$

Now, we estimate G_2 . Here, $n > M\epsilon^{-2}$, so

$$\frac{|ES'_n|}{\sqrt{n \log n}(x + \epsilon)} < \frac{E\xi_1^2}{(\log n)(x + \epsilon)^2} < \frac{E\xi_1^2}{M} < \epsilon, \quad \text{for } M \rightarrow \infty. \quad (2.33)$$

We have

$$G_2 \leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} (x + \epsilon) \left[P \left(\sup_i |a_{ni} \xi_i| > \sqrt{n \log n} (x + \epsilon) \right) + P \left(|S'_n - ES'_n| \geq \frac{\sqrt{n \log n} (x + \epsilon)}{2} \right) \right] dx \quad (2.34)$$

$$=: G_{21} + G_{22}.$$

We estimate G_{21} first. Similar to the proof of (2.16), we have

$$\begin{aligned} G_{21} &\leq CE |\xi_1| \int_0^{\infty} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\ &\quad \times \sum_{k=n}^{\infty} I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \\ &\leq CE |\xi_1| \int_0^{\infty} \sum_{k=d(\epsilon)+1}^{\infty} I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \\ &\quad \times \sum_{n=d(\epsilon)+1}^k \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\ &\leq CE \xi_1^2 \int_0^{\infty} \sum_{k=d(\epsilon)+1}^{\infty} (\log n)^{\delta-1} (x + \epsilon)^{-1} \\ &\quad \times I \left\{ \sqrt{k \log k} (x + \epsilon) < |\xi_1| \leq \sqrt{(k+1) \log(k+1)} (x + \epsilon) \right\} dx \\ &\leq CE \xi_1^2 \int_0^{\infty} |\log |\xi_1| - \log (x + \epsilon)|^{(\delta-1)} (x + \epsilon)^{-1} I \{ |\xi_1| > (x + \epsilon) \} dx \\ &\leq CE \xi_1^2 |\log^+ |\xi_1| - \log \epsilon|^{\delta} \leq CE \xi_1^2 (\log^+ |\xi_1|)^{\delta} + CE \xi_1^2 (-\log \epsilon)^{\delta}, \end{aligned} \quad (2.35)$$

then

$$\lim_{M \rightarrow \infty} \limsup_{\epsilon \searrow 0} \epsilon^{2\delta} G_{21} = 0. \quad (2.36)$$

By Lemma 1.6, for $q > 2$, we have

$$\begin{aligned}
 G_{22} &= \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} (x+\epsilon) P \left\{ |S'_n - ES'_n| \geq \frac{\sqrt{n \log n}(x+\epsilon)}{2} \right\} dx \\
 &\leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
 &\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} \left\{ \left(\sum_{i=-\infty}^{\infty} E(a_{ni}\xi_1)^2 I \left\{ |a_{ni}\xi_i| \leq \sqrt{n \log n}(x+\epsilon) \right\} \right)^{q/2} \right. \\
 &\quad \left. + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I \left\{ |a_{ni}\xi_i| \leq \sqrt{n \log n\epsilon} \right\} \right. \\
 &\quad \left. + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I \left\{ \sqrt{n \log n\epsilon} < |a_{ni}\xi_i| \leq \sqrt{n \log n}(x+\epsilon) \right\} \right\} dx \\
 &=: G_{221} + G_{222} + G_{223}.
 \end{aligned} \tag{2.37}$$

For G_{221} , we have

$$\begin{aligned}
 G_{221} &\leq \sum_{n=d(\epsilon)+1}^{\infty} \int_0^{\infty} \frac{(\log n)^{\delta-q/2}}{n} (x+\epsilon)^{1-q} \left(E\xi_1^2 I \{ |a_{ni}\xi_1| \leq n(x+\epsilon) \} \right)^{q/2} dx \\
 &\leq C\epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n} \leq CM^{\delta+1-q/2} \epsilon^{-2\delta}.
 \end{aligned} \tag{2.38}$$

Next, turning to G_{222} , it follows that

$$\begin{aligned}
 G_{222} &\leq \epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
 &\quad \times \sum_{j=1}^{\infty} (\#I_{nj}) j^{-q} \left\{ \sum_{k=2}^{2n} E|\xi_1|^q I \left\{ \sqrt{k \log k\epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1)\epsilon} \right\} \right. \\
 &\quad \left. + \sum_{k=2n+1}^{(j+1)n} E|\xi_1|^q I \left\{ \sqrt{k \log k\epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1)\epsilon} \right\} \right\} \\
 &=: G_{2221} + G_{2222},
 \end{aligned} \tag{2.39}$$

then

$$\begin{aligned}
G_{2221} &\leq C\epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} \sum_{k=2}^{2n} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq C\epsilon^{2-q} \sum_{k=2}^{\infty} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \sum_{n=[k/2]}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} \\
&\leq C\epsilon^{2-q} \sum_{k=2}^{\infty} \frac{(\log k)^{\delta-q/2}}{k^{q/2-1}} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq C \sum_{k=2}^{\infty} (\log k)^{\delta-1} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq C \sum_{k=2}^{\infty} (\log k)^{-1} |\log^+ |\xi_1| - \log \epsilon|^\delta E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq CE\xi_1^2 (\log^+ |\xi_1|)^\delta + CE\xi_1^2 (-\log \epsilon)^\delta.
\end{aligned} \tag{2.40}$$

For G_{2222} , it follows that

$$\begin{aligned}
G_{2222} &\leq C\epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
&\quad \times \sum_{k=2n+1}^{\infty} \sum_{j \geq k/n-1} (\#I_{nj}) j^{-q} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq C\epsilon^{2-q} \sum_{k=d(\epsilon)+1}^{\infty} k^{1-q} E|\xi_1|^q I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \sum_{n=d(\epsilon)+1}^{[k/2]} \frac{(\log n)^{\delta-q/2}}{n^{1-q/2}} \\
&\leq C \sum_{k=d(\epsilon)+1}^{\infty} (\log k)^{\delta-1} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq C \sum_{k=d(\epsilon)+1}^{\infty} (\log k)^{-1} |\log^+ |\xi_1| - \log \epsilon|^\delta E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} \leq |\xi_1| < \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\leq CE\xi_1^2 (\log^+ |\xi_1|)^\delta + CE\xi_1^2 (-\log \epsilon)^\delta.
\end{aligned} \tag{2.41}$$

Finally, $q > 2$, we have

$$\begin{aligned}
G_{223} &\leq C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} E|\xi_1|^q I \left\{ |\tilde{a}\xi_1| > \sqrt{n \log n \epsilon} \right\} \\
&\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} \sum_{j=1}^{\infty} (\#I_{nj}) j^{-q} \left\{ \sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n} \right\} \\
&\quad \times I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1) (x+\epsilon)} \right\} dx \\
&\leq C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} E|\xi_1|^q I \left\{ |\tilde{a}\xi_1| > \sqrt{n \log n \epsilon} \right\} \\
&\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} I \left\{ |\xi_1| \leq \sqrt{2n \log(2n) (x+\epsilon)} \right\} dx + C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1-q/2}} E|\xi_1|^q \\
&\quad \times \int_0^{\infty} \sum_{k=2n+1}^{\infty} \frac{k^{1-q}}{(x+\epsilon)^{q-1}} I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1) (x+\epsilon)} \right\} dx \\
&\leq C \sum_{k=d(\epsilon)+1}^{\infty} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} < |\xi_1| \leq \sqrt{(k+1) \log(k+1) \epsilon} \right\} \sum_{n=d(\epsilon)+1}^k \frac{(\log n)^{\delta-1}}{n} \\
&\quad + CE\xi_1^2 \int_0^{\infty} (x+\epsilon)^{-1} \\
&\quad \times \sum_{k=d(\epsilon)+1}^{\infty} (\log k)^{\delta-1} I \left\{ \sqrt{k \log k (x+\epsilon)} < |\xi_1| \leq \sqrt{(k+1) \log(k+1) (x+\epsilon)} \right\} dx \\
&\leq C \sum_{k=2}^{\infty} (\log k)^{\delta} E\xi_1^2 I \left\{ \sqrt{k \log k \epsilon} < |\xi_1| \leq \sqrt{(k+1) \log(k+1) \epsilon} \right\} \\
&\quad + CE\xi_1^2 \int_0^{\infty} |\log^+ |\xi_1| - \log(x+\epsilon)|^{(\delta-1)} (x+\epsilon)^{-1} I \{ |\xi_1| > (x+\epsilon) \} dx \\
&\leq CE\xi_1^2 |\log^+ |\xi_1| - \log \epsilon|^{\delta} \leq CE\xi_1^2 (\log^+ |\xi_1|)^{\delta} + CE\xi_1^2 (-\log \epsilon)^{\delta}.
\end{aligned} \tag{2.42}$$

From (2.38) to (2.42), we can get

$$\lim_{M \rightarrow \infty} \lim_{\epsilon \searrow 0} \epsilon^{2\delta} G_{22} = 0. \tag{2.43}$$

(2.29) can be derived by (2.32), (2.36), and (2.43). \square

Proof of Theorem 1.2. Without loss of generality, we set $\tau = 1$. It is easy to see that

$$\begin{aligned} & \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta-1}}{n^2 \log n} ES_n^2 I \left\{ |S_n| \geq \sqrt{n \log \log n \epsilon} \right\} \\ &= \epsilon^2 \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta}}{n \log n} P \left\{ |S_n| \geq \sqrt{n \log \log n \epsilon} \right\} \\ &+ \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta-1}}{n^2 \log n} \int_{\sqrt{n \log \log n \epsilon}}^{\infty} 2xP(|S_n| \geq x) dx. \end{aligned} \quad (2.44)$$

So, we only prove the following two propositions:

$$\epsilon^{2\delta+2} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta}}{n \log n} P \left\{ |S_n| \geq \sqrt{n \log \log n \epsilon} \right\} = \frac{E|N|^{2(\delta+1)}}{\delta+1}, \quad (2.45)$$

$$\epsilon^{2\delta} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta-1}}{n^2 \log n} \int_{\sqrt{n \log \log n \epsilon}}^{\infty} 2xP(|S_n| \geq x) dx = \frac{E|N|^{2(\delta+1)}}{\delta(\delta+1)}. \quad (2.46)$$

The proof of (2.45) can be referred to [6], and the proof of (2.46) is similar to Propositions 2.1 and 2.2. \square

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